# Sheet 12

$\Omega \subset \mathbb{R}^n$ open

① Fundamental lemma of the Calculus of Variations (Tutorial 8)

If $u \in L^1_{loc}(\Omega)$, $\int_\Omega u\varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$

then $u = 0$ (a.e.).

(a) Suppose $u \in L^1_{loc}(\Omega)$, and $v_1, v_2$ are both weak derivatives of $u$ wrt $x_i$. So

$$-\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = \int_\Omega \varphi v_1 \, dx = \int_\Omega \varphi v_2 \, dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

So $\int_\Omega \varphi (v_1 - v_2) \, dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$

By FL of the COV, $v_1 = v_2$ a.e. □

(b) Claim: Suppose, for $u \in L^1_{loc}(\Omega)$, $\int_\Omega u\varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$ such that $\int_\Omega \varphi = 0$. Then $u$ is constant.

Proof: Take any ~~$f \in L^1(\Omega)$ such that~~ $f \in C_c^\infty(\Omega)$ such that $\int_\Omega f \, dx = 1$. Let $\varphi \in C_c^\infty(\Omega)$. Now define

$$\psi(x) := \varphi(x) - \left(\int_\Omega \varphi\right) f(x), \quad x \in \Omega.$$

Then $\psi \in C_c^\infty(\Omega)$, and $\int_\Omega \psi(x)\, dx = \int_\Omega \varphi - \left(\int_\Omega \varphi\right)\left(\int_\Omega f\right)$
$= 0$.

So, by assumption, $\int_\Omega u\psi = 0$.

i.e. $$\int_\Omega u(x)\varphi(x) - u(x) f(x) \left(\int_\Omega \varphi(y)\, dy\right) dx = 0$$

But $$\int_\Omega u(x) f(x) \left(\int_\Omega \varphi(y) \, dy\right) dx$$

$$= \int_\Omega \varphi(x) \int_\Omega u(y) f(y)\, dy \, dx$$

So we have

$$\int_\Omega \left( u(x) - \left( \int_\Omega u(y) f(y)\, dy \right) \right) \varphi(x) = 0$$

True for all $\varphi \in C_c^\infty(\Omega)$

By FL of CofV, $u(x) - \int_\Omega uf\, dy = \text{constant}$ a.e. $\square$

③ $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$, $E \subset \Omega$ open.

Claim: $u \in W^{1,p}(E)$, weak derivative of $u$ on $E$ = weak deriv. of $u$ on $\Omega$ restricted to $E$.

Let $\varphi \in C_c^\infty(E)$. Extend by zero to all of $\mathbb{R}^n$. So $\varphi \in C_c^\infty(\Omega)$.
Let $1 \le i \le n$, and let $\frac{\partial u}{\partial x_i}$ denote the weak derivative of $u$ on $\Omega$ wrt $x_i$.

Then
$$\int_E u \frac{\partial \varphi}{\partial x_i}\, dx = \int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx \quad (\varphi = 0 \text{ on } \Omega \setminus E)$$
$$= -\int_\Omega \frac{\partial u}{\partial x_i} \varphi\, dx \quad \left(\frac{\partial u}{\partial x_i} \text{ weak deriv of } u \text{ on } \Omega\right)$$
$$= -\int_E \frac{\partial u}{\partial x_i} \varphi\, dx \quad (\varphi = 0 \text{ on } \Omega \setminus E)$$

So $\frac{\partial u}{\partial x_i}$ weak derivative of $u$ on $E$ too.

Clearly $\int_E \left|\frac{\partial u}{\partial x_i}\right|^p dx \le \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx$

$\left( \text{or } \left\|\frac{\partial u}{\partial x_i}\right\|_{L^\infty(E)} \le \left\|\frac{\partial u}{\partial x_i}\right\|_{L^\infty(\Omega)} \right)$ so $u \in W^{1,p}(E)$

④ (First part: Tutorial 2)

$B$ = unit ball in $\mathbb{R}^2$. $0 < s < 1$

$u(x) = \frac{1}{|x|^s}$

(i) $$\int_B |u(x)|^p \, dx = \underset{\omega_2}{\overset{\shortparallel}{2\pi}} \int_0^1 r \left(\frac{1}{r^s}\right)^p dr$$

$$= 2\pi \int_0^1 r^{1-sp} \, dr$$

$< \infty$ iff $1 - sp > -1$ ie $p < 2/s$.

So $u \in L^p(B)$ for $1 \leq p < 2/s$.

(ii) $u(x,y) = (x^2+y^2)^{-s/2}$ $u$ strongly diff'ble provided $(x,y) \neq (0,0)$.

So $$\frac{\partial u}{\partial x} = \frac{-s}{2} \cdot 2x \, (x^2+y^2)^{-\frac{s}{2}-1}$$

$$= -s\,x\,(x^2+y^2)^{-\left(\frac{s+2}{2}\right)}$$

$$= \frac{-sx}{|(x,y)|^{s+2}}.$$ Similarly $$\frac{\partial u}{\partial y} = \frac{-sy}{|(x,y)|^{s+2}}$$

(iii) Note that $$\left|\frac{\partial u}{\partial x}(x,y)\right| = \left|\frac{-sx}{|(x,y)|^{s+2}}\right| \leq \frac{|s|\,|(x,y)|}{|(x,y)|^{s+2}}$$

$$\leq \frac{1}{|(x,y)|^{s+1}}$$

Hence, $$\int_B \left|\frac{\partial u}{\partial x}\right|^p d(x,y) \leq 2\pi \int_0^1 r \left(\frac{1}{r^{s+1}}\right)^p dr$$

$$= 2\pi \int_0^1 r^{1-p(s+1)} \, dr$$

$< \infty$ iff $1 - p(s+1) > -1$

ie iff $p < \frac{2}{s+1}$

So $\frac{\partial u}{\partial x} \in L^p(B)$ for $1 \leq p < \frac{2}{s+1}$ (sharp: harder)

For $\frac{\partial u}{\partial y}$ likewise.

(iv) <u>We are not done yet!</u> Still need to show $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are weak derivatives of $u$ <u>on all of $B$</u> (we can make them whatever we like at $(0,0)$)

i.e. need to show

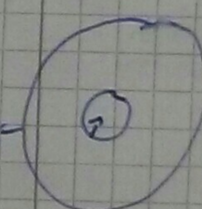
$$(*) \quad \int_B u \frac{\partial \varphi}{\partial x} \, d(x,y) = -\int_B \frac{\partial u}{\partial x} \varphi \, d(x,y) \quad \forall \varphi \in C_c^\infty(B).$$

Let $\varepsilon \in (0,1)$ be small and split $B = B_\varepsilon \cup B \setminus B_\varepsilon$.

By Gauss-Green / Integration by parts $\left( u, \varphi \in C^\infty(\overline{B \setminus B_\varepsilon}) \right)$

$$\int_{B \setminus B_\varepsilon} u \frac{\partial \varphi}{\partial x} \, d(x,y) = -\int_{B \setminus B_\varepsilon} \underbrace{\frac{\partial u}{\partial x_1}}_{\text{strong deriv}} \varphi \, dx + \int_{\partial B_\varepsilon} \varphi u \overbrace{\nu^{(1)}}^{\text{unit normal}} \, dS$$

$$+ \underbrace{\int_{\partial B} \varphi u \nu^{(1)} \, dS}_{=0} \qquad \varphi = 0 \text{ on } \partial B.$$



So

$$(2) \quad \int_B u \frac{\partial \varphi}{\partial x} \, d(x,y) = \underbrace{\int_{B_\varepsilon} \frac{\partial \varphi}{\partial x} u \, dx}_{\alpha_\varepsilon} - \underbrace{\int_{B \setminus B_\varepsilon} \frac{\partial u}{\partial x} \varphi \, dx}_{\beta_\varepsilon} + \underbrace{\int_{\partial B_\varepsilon} u \varphi \underbrace{\frac{-x}{|(x,y)|}}_{\text{- normal}} \, dS}_{\gamma_\varepsilon}$$

$$|\alpha_\varepsilon| \leq 2\pi \left\| \frac{\partial \varphi}{\partial x} \right\|_\infty \int_0^\varepsilon r^{1-s} \, dr \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

$$|\gamma_\varepsilon| = \left| \int_{\partial B_\varepsilon} |(x,y)|^{-s} \varphi(x,y) \frac{-x}{|(x,y)|} \, dS \right| \leq \|\varphi\|_\infty \, \varepsilon^{1-s} \, 2\pi \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

For $\beta_\varepsilon$: $\frac{\partial u}{\partial x} \in L^1(B)$ (part (iii)), $\varphi \in L^\infty(B)$

Let $v_\varepsilon(x) = \chi_{B\setminus B_\varepsilon}(x)\, \frac{\partial u}{\partial x}(x)\, \varphi(x)$

$$v_\varepsilon(x) \longrightarrow \frac{\partial u}{\partial x}(x)\, \varphi(x) \quad \forall x \in B\setminus\{0\}.$$

Also $|v_\varepsilon(x)| \le \left|\frac{\partial u}{\partial x}(x)\right| \|\varphi\|_\infty \in L^1(B)$

Hence, by CDCT,

$$\beta_\varepsilon = \int_{B\setminus B_\varepsilon} \frac{\partial u}{\partial x}\, \varphi\, dx = \int_B v_\varepsilon\, dx \xrightarrow{\varepsilon \to 0} \int_B \frac{\partial u}{\partial x}\, \varphi$$

Hence, taking limit as $\varepsilon \to 0$ in (2), we have

$$\int_B \frac{\partial \varphi}{\partial x}\, u = -\int_B \cancel{\frac{\partial \varphi}{\partial x}}\, \frac{\partial u}{\partial x}\, \varphi$$ , as required.

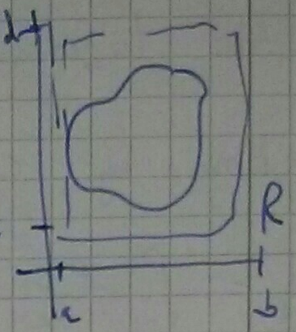
For $\frac{\partial u}{\partial y}$ - same.

(v) We have shown that the strong derivatives of $u$ calculated in (ii) are in $L^p(B)$ for $1 \le p < \frac{2}{r+1}$, and are weak derivatives of $u$ in $B$. So $u \in W^{1,p}(B)$.

② Suppose $f, g \in L^1_{loc}(\mathbb{R})$ are weakly diff'ble.

Define $u(x,y) := f(x) + g(y)$ $\quad (x,y) \in \mathbb{R}^2$

Then $u \in L^1_{loc}(\mathbb{R}^2)$. Why? Let $E \subset\subset \mathbb{R}^2$.




Then there exists a rectangle $(a,b) \times (c,d)$ in $\mathbb{R}^2$ such that $E \subseteq (a,b) \times (c,d)$

So
$$\int_E |u(x,y)| \, d(x,y) \leq \int_R |u(x,y)| \, d(x,y)$$
$$\leq \int_R |f(x)| + |g(y)| \, d(x,y)$$
$$= \int_d^c \int_a^b |f(x)| \, dx \, dy + \int_d^c \int_a^b |g(y)| \, dx \, dy$$
$$= (d-c) \int_a^b |f| + (b-a) \int_c^d |g| < \infty \quad (f, g \text{ locally integrable})$$

So $u \in L^1_{loc}(\mathbb{R}^2)$

Claim: $u$ is weakly diff'ble, with
$$\frac{\partial u}{\partial x}(x,y) = f'(x) \qquad \frac{\partial u}{\partial y}(x,y) = g'(y).$$

Proof: Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. Then, as above, take a rectangle $R = (a,b) \times (c,d) \supset\supset \mathrm{supp}(\varphi)$.

So $\forall y \in \mathbb{R}$: $x \longmapsto \varphi(x,y) \in C_c^\infty(a,b)$

$\forall x \in \mathbb{R}$: $y \longmapsto \varphi(x,y) \in C_c^\infty(c,d)$

Now note:

$$\int_{\mathbb{R}^2} \frac{\partial \varphi}{\partial x}(x,y)\, u(x,y)\, d(x,y)$$ (Integrand in $L^1$ since, so apply Fubini)

$$= \int_{R} \frac{\partial \varphi}{\partial x}(x,y)\, (f(x) + g(y))\, d(x,y)$$

$$= \underbrace{\int_c^d \int_a^b \frac{\partial \varphi}{\partial x}(x,y)\, f(x)\, dx\, dy}_{A} + \underbrace{\int_c^d \int_a^b \frac{\partial \varphi}{\partial x}(x,y)\, g(y)\, dx\, dy}_{B}$$

$$A = \int_c^d \int_a^b \frac{\partial \varphi}{\partial x}(x,y)\, f(x)\, dx\, dy$$

Fix $y$: $\displaystyle\int_a^b \frac{\partial \varphi}{\partial x}(x,y)\, f(x)\, dx = \int_a^b h'(x) f(x)\, dx$

where $h(x) = \varphi(x,y)$ $\displaystyle = -\int_a^b h(x) f'(x)\, dx$ $f'$ weak deriv of $f$.

$$= -\int_a^b \varphi(x,y)\, f'(x)\, dx.$$

So $\displaystyle A = -\int_c^d \int_a^b \varphi(x,y)\, f'(x)\, dx\, dy = -\int_{\mathbb{R}^2} \varphi(x,y)\, f'(x)\, d(x,y)$

Consider B:
Fix $y$: $\displaystyle\int_a^b \frac{\partial \varphi}{\partial x}(x,y)\, g(y)\, dx = g(y) \int_a^b \frac{\partial \varphi}{\partial x}(x,y)\, dx$

$$\overset{\text{FTC}}{=} g(y)\, (\underbrace{\varphi(b,y) - \varphi(a,y)}_{0}) = 0$$

$\operatorname{supp} \varphi \subset\subset (a,b) \times (c,d)$

So $B = 0$.

Hence $\displaystyle\int_{\mathbb{R}^2} \frac{\partial \varphi}{\partial x}\, u\, d(x,y) = -\int_{\mathbb{R}^2} \varphi(x,y)\, f'(x)\, d(x,y)$

$(x,y) \longmapsto f'(x)$ is in $L^1_{loc}(\mathbb{R}^2)$ (easy to show)

So ~~f'~~ this is the weak deriv of $u$ wrt $x$.

$\frac{\partial u}{\partial y}(x,y) = g'(y)$ : exactly same argmt.

$\square$